On the Most Weight w Vectors in a dimension k Binary Code

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Abstract

Ahlswede, Aydinian, and Khachatrian posed the following problem: what is the maximum number of Hamming weight w vectors in a k-dimensional subspace of $\mathbb{F}_2^n$? The answer to this question could be relevant to coding theory, since it sheds light on the weight distributions of binary linear codes. We give some partial results. We also provide a conjecture for the complete solution when w is odd as well as for the case $k \geq 2w$ and w even.

One tool used to study this problem is a linear map that decreases the weight of nonzero vectors by a constant. We characterize such maps.

1 Introduction

Ahlswede, Aydinian, and Khachatrian [1] introduced extremal problems with *dimension constraints*. Begin with a class of set systems on the ground set $[n] = \{1, 2, \ldots, n\}$. For example, the set of intersecting families on $[n]$. Given a field $\mathbb{F}$, a set system in this class can be viewed as a collection of $\{0,1\}$-valued vectors in $\mathbb{F}^n$. The extremal problem with a dimension constraint is to find the largest set system that has rank at most k.

In this paper, we consider a dimension constraint on uniform hypergraphs. To be more precise, first recall that the Hamming weight of a vector v, denoted $wt(v)$, is the number of entries of v that are nonzero. Given $n, k, w \in \mathbb{N}$ and a field $\mathbb{F}$, denote $M_{\mathbb{F}}(n, k, w)$ to be the maximum number of $\{0,1\}$-valued vectors with Hamming weight w in a k-dimensional subspace of $\mathbb{F}^n$. Ahlswede, Aydinian, and Khachatrian found a formula for $M_{\mathbb{R}}$ [1].

Theorem 1 (Ahlswede, Aydinian, and Khachatrian). *Given $n, k, w \in \mathbb{N}$,*

$$M_{\mathbb{R}}(n, k, w) = M_{\mathbb{R}}(n, k, n - w),$$

and for $w \le n/2$,

$$M_{\mathbb{R}}(n,k,w) = \begin{cases} \binom{k}{w} & \text{if } 2w \le k; \\ \binom{2(k-w)}{k-w} 2^{2w-k} & \text{if } k < 2w < 2(k-1); \\ 2^{k-1} & \text{if } k-1 \le w. \end{cases}$$

This paper focusses on the case $\mathbb{F} = \mathbb{F}_2$. Given $n, k, w \in \mathbb{N}$, denote

$$m(n,k,w) = M_{\mathbb{F}_2}(n,k,w).$$

A complete description of $m(n,k,w)$ might be relevant to coding theory, since it would shed light on the weight distributions of binary linear codes. Determining $m(n,k,w)$ requires different techniques from those used to determine $M_{\mathbb{R}}(n,k,w)$. In particular, the proof in [1] of the $\mathbb{R}^n$ case makes explicit use of the fact that the sum of a non-empty collection of positive numbers in $\mathbb{R}$ is nonzero.

In [2] Ashikhmin, Cohen, Krivelevich, and Litsyn give some upper bounds for $m(n,k,w)$ and cite some conjectures for $m(n,k,w)$ from personal correspondence with Khachatrian. In [1] it is noted that $m(n,k,w)$ depends crucially on the parity of w, while $M_{\mathbb{R}}(n,k,w)$ does not. In particular, every k-dimensional subspace of $\mathbb{F}_2^n$ has $2^k - 1$ nonzero elements, and either 0 or 2^{k-1} odd weight elements. Thus $m(n,k,w) \le 2^k - 1$ if w is even, and $m(n,k,w) \le 2^{k-1}$ if w is odd. If equality holds in the even case, then there is a dimension k subspace of $\mathbb{F}_2^n$ all of whose nonzero vectors have weight w, which we call an *equidistant linear code*. The following are noted in [1] and are consequences of standard facts about equidistant linear codes over $\mathbb{F}_2$.

Proposition 2. *Given $n, k, w \in \mathbb{N}$ we have $m(n,k,w) = 2^k - 1$ if and only if there is some $t \in \mathbb{N}$ for which $w = t2^{k-1}$ and $n \ge t(2^k - 1) = 2w - t$.*

Proposition 3. *Suppose w is odd. We have $m(n,k,w) = 2^{k-1}$ if and only if $k \le w + 1$ and $n \ge w + k - 1$.*

We generalize these results. Given $w \in \mathbb{N}$, denote

$$f_2(w) = \max \{e \in \mathbb{N} \,:\, 2^e \text{ divides } w\}.$$

In Section 2, we prove the following.

Theorem 4. *Given $n, k, w \in \mathbb{N}$, we have*

$$m(n,k,w) \le 2^k - 2^{(k-1)-f_2(w)},$$

with equality if and only if there exists $t \in \mathbb{N}$ such that $t \ge (k-1) - f_2(w) \ge 0$, $w = t2^{f_2(w)}$, and $n \ge 2w - t + (k-1) - f_2(w)$.

To prove this theorem, we need to understand the structure of equidistant linear codes. This is provided by a theorem of Bonisoli [3], which needs the concepts of monomial equivalence and the binary simplex code. More can be found about these concepts in, for example, [5].

Given $n \in \mathbb{N}$, a field $\mathbb{F}$, and subspaces $V, W \subseteq \mathbb{F}^n$, we say a linear map $\phi : V \to W$, is a *monomial equivalence* if ϕ is bijective and there are $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}^\times = \mathbb{F} \setminus \{\vec{0}\}$ and a permutation $\sigma : [n] \to [n]$ such that for all $v = (v_1, v_2, \ldots, v_n) \in V$, we have

$$\phi(v) = \left(\lambda_1 v_{\sigma(1)}, \lambda_2 v_{\sigma(2)}, \ldots, \lambda_n v_{\sigma(n)}\right).$$

Define the *binary simplex code* of dimension k, denoted $\mathcal{S}_k$, to be the row span of M_k, the $k \times (2^k - 1)$ matrix whose columns are the unique vectors of $\mathbb{F}_2^k \setminus \{\vec{0}\}$. It is not difficult to show that $\mathcal{S}_k$ is a k-dimensional equidistant code whose nonzero codewords have weight 2^{k-1}. Proposition 2 claims that if $w = t2^{k-1}$ and $n \geq 2w - t$ then there is an equidistant linear code of dimension k and weight w in $\mathbb{F}_2^n$. Indeed, define

$$\begin{array}{cc} & \overbrace{\qquad\qquad\qquad}^{\leftarrow \quad t \text{ times} \quad \rightarrow} \quad \overbrace{\qquad\qquad\qquad}^{\leftarrow n - t(2^k - 1) \text{ columns} \rightarrow} \\ M_{k,t,n} = & \left[\begin{array}{c|c|c|c|c} M_k & M_k & \cdots & M_k & \qquad\qquad 0 \qquad\qquad \end{array}\right]. \end{array}$$

Define $\mathcal{S}(k, t, n)$ to be the row span of this matrix. It is clear that $\mathcal{S}(k, t, n) \subseteq \mathbb{F}_2^n$, $\dim \mathcal{S}(k, t, n) = k$, and $\mathcal{S}(k, t, n)$ has constant nonzero weight w.

Proposition 5 (Bonisoli [3])**.** *Let $n, k, w \in \mathbb{N}$. If $V \subseteq \mathbb{F}_2^n$ is a k-dimensional equidistant linear code of weight w then $w = t2^{k-1}$ and V is monomially equivalent to $\mathcal{S}(k, t, n)$.*

Clearly, every monomial equivalence is a weight-preserving linear map (i.e. a linear map that does not change the Hamming weight of any vector in its domain). The converse is a theorem of MacWilliams [7].

Theorem 6 (The MacWilliams Extension Theorem [7])**.** *Let $n \in \mathbb{N}$, let $\mathbb{F}$ be a finite field, and let $V, W \subseteq \mathbb{F}^n$ be subspaces. If $\phi : V \to W$ is a bijective weight-preserving linear map then ϕ is a monomial equivalence.*

We prove a generalization of this theorem.

Definition 1. Let $n, c \in \mathbb{N}$, let $\mathbb{F}$ be a finite field, and let $V, W \subseteq \mathbb{F}^n$ be subspaces. We say a linear map $\phi : V \to W$ is a *c-killer* if for all $v \in V \setminus \{\vec{0}\}$,

$$\mathrm{wt}(\phi(v)) = \mathrm{wt}(v) - c.$$

Given $n \in \mathbb{N}$, $A \subseteq [n]$, and field $\mathbb{F}$, denote the *coordinate projection onto the coordinates* A by $\pi_A : \mathbb{F}^n \to \mathbb{F}^n$. That is, π_A is the identity on A and the 0 map on the complement of A.

Theorem 7. *Let $V, W \subseteq \mathbb{F}^n$ be subspaces. If $\phi : V \to W$ is a c-killer, then ϕ is a monomial equivalence composed with a coordinate projection.*

We use Theorem 7 to show the following.

Proposition 8. *Let $n, k, w \in \mathbb{N}$, where $1 \leq k, w \leq n$ and w is odd. If $m(n,k,w) = 2^{k-1} - 1$, then $k \leq 3$. Furthermore, we have*

- $m(n,1,w) = 2^{1-1} - 1 = 0$ *is impossible.*
- $m(n,2,w) = 2^{2-1} - 1 = 1$ *if and only if* $n = w$.
- $m(n,3,w) = 2^{3-1} - 1 = 3$ *if and only if* $w = 1$ *or* $n = w + 1$.

We finish the paper with some conjectures and evidence for those conjectures. In particular, we conjecture that for w odd,

$$m(n,k,w) = M_{\mathbb{R}}(n,k,w)$$

(despite the fact that equality is false in the case w is even). We also conjecture that for w even and $k \geq 2w$,

$$m(n,k,w) = \binom{k+1}{w}. \tag{1.1}$$

In [2] there a reference to a personal correspondence in which Khachatrian also conjectures (1.1). Khachatrian also conjectures that for $w < k < 2w$, w even and k odd,

$$m(n,k,w) = 2^{2w-k}\binom{2k-2w}{k-w} + \sum_{i=0}^{k-w} \binom{2k-2w}{2i}\binom{2w-k}{\frac{w-2i}{2}}.$$

The paper is organized as follows. In Section 2 we prove Theorem 4. In Section 3 we prove Theorem 7. In Section 3.4 we use the killer classification to prove Proposition 8. In Section 4 we give evidence for our conjectures.

2 The bound on $m(n,k,w)$

2.1 A supporting Theorem and Lemma

For q a prime power, denote the field with q elements by $\mathbb{F}_q$. We will use the following theorem of Bose and Burton [4]

Theorem 9 (Bose and Burton [4])**.** *In an $\mathbb{F}_q$-vector space V, let S be a set of nonzero vectors that meets every subspace of a given dimension b. Then $|S| \geq (q^{k-b+1} - 1)/(q-1)$ with equality iff S consists of the nonzero points (non-collinear vectors) in a subspace of dimension $k - b + 1$.*

One way to meet the bound in Theorem 4 is to construct a space where the non-weight-w vectors form a subspace of dimension $k - 1 - f_2(w)$. We use the following lemma to establish that there is such a space under the conditions of Theorem 4.

Lemma 10. *Let $n, k, w, l \in \mathbb{N}$ where $l \leq k$. There is a k-dimensional code $V \subseteq \mathbb{F}_2^n$ whose non-weight-w vectors are contained in a subspace of dimension l if and only if there is an integer $t \geq l$ such that $w = t2^{k-l-1}$ and $n \geq 2w - t + l$.*

Proof. Suppose there is $t \geq l$ such that $w = t2^{k-l-1}$ and $n \geq 2w - t + l$. Then define M'_{k-l} to be M_{k-l} with a zero column appended on the right. Define V' to be the row space of the $k \times (2w - t + l)$ matrix

$$G' = \left[\begin{array}{c|c|c|c|c|c|c} 11\cdots1 & 0 & & 0 & 0 & & 0 \\ \cline{1-2} 0 & 11\cdots1 & \cdots & & & \cdots & \\ \cline{2-2} & 0 & & 11\cdots1 & & & \\ \hline M'_{k-l} & M'_{k-l} & & M'_{k-l} & M_{k-l} & & M_{k-l} \end{array}\right].$$

To be explicit, the number of M'_{k-l} blocks is l, and the number of M_{k-l} blocks is $t - l$. The number of columns of G' is therefore $l2^{k-l} + (t-l)(2^{k-l} - 1) = 2w - t + l \leq n$. Thus, we may pad G' with sufficiently many 0 columns to get G, a $k \times n$ generator matrix. Let V be the vector space generated by G. Suppose $v \in V$ is the sum of a subset of the rows of G, at least one of which is among the bottom $k - l$ rows. Denote the complement of a binary vector u by $\overline{u}$. There is $s \in \mathcal{S}_{k-l} \setminus \{\vec{0}\}$ such that, up to permutation of the first l blocks, $v = (\overline{s0}, \ldots, \overline{s0}, s0, \ldots, s0, s, \ldots, s, 0, \ldots, 0)$. But $\mathrm{wt}(s) = 2^{k-l-1}$, and $\mathrm{wt}(\overline{s0}) = 2^{k-l-1}$, so the total weight of v is $\mathrm{wt}(v) = (l + (t-l))2^{k-l-1} = w$. Thus all non-weight-$w$ vectors are in the span of the first l rows, and we have found the desired vector space.

For the other direction, let V be a k-dimensional subspace of $\mathbb{F}_2^n$ whose non-weight-w vectors are in a subspace U of dimension l. Notice that $|U \setminus \{\vec{0}\}| = 2^l - 1 = 2^{k-(k-l+1)+1} - 1 < 2^{k-(k-l)+1} - 1$. By Theorem 9, there is a subspace $\mathcal{C}$ of $V \setminus (U \setminus \{\vec{0}\})$ of dimension $k - l$. This code has constant distance w, so $w = t2^{k-l-1}$ for some integer t.

The *support* of a set of vectors V, denoted $s(V)$, is the set of coordinates on which V is not always zero. Define Z to be the complement of the support of $\mathcal{C}$. We claim that the projection π_Z is injective as a function of U. Pick $u \in U \setminus \{\vec{0}\}$. Given any vector $c \in \mathcal{C} \setminus \{\vec{0}\}$, we have $c + u \notin U$ and hence $wt(c + u) = w$. Thus

$$\mathrm{wt}(\pi_{s(u)}(c)) = \frac{1}{2}\mathrm{wt}(u). \tag{2.1}$$

In particular, $\pi_{s(u)}(\mathcal{C})$ is an equidistant linear code of dimension $\dim \mathcal{C} = k - l$. By Proposition 5 we have that, up to permutation of entries, $\pi_{s(u)}(\mathcal{C}) = \mathcal{S}(k-l, \mathrm{wt}(u)/2^{k-l}, \mathrm{wt}(u))$. In particular,

$$\mathrm{wt}(\pi_Z(u)) = |s(u) \setminus s(\pi_{s(u)}(\mathcal{C}))| = \mathrm{wt}(u) - (2^{k-l} - 1)\,\mathrm{wt}(u)/2^{k-l} = \mathrm{wt}(u)/2^{k-l} > 0 \tag{2.2}$$

Thus π_Z is injective on U and $\dim \pi_Z(U) = \dim U = l$. Thus $|s(\pi_Z(U))| \geq l$. But then $n \geq |s(V)| = |s(C)| + |s(\pi_Z(U))| \geq 2w - t + l$.

We have left to show that $t \geq l$. If $T \subseteq \mathbb{F}_2^n$ is a dimension l subspace then T has a vector of weight at least l. Choose $u' \in \pi_Z(U)$ with $\mathrm{wt}(u') \geq l$. Let u be the corresponding element in U. By (2.2), $wt(u)/2^{k-l} = wt(u') \geq l$, so $wt(u) \geq l2^{k-l}$. For $c \in \mathcal{C} \setminus \{\vec{0}\}$ we have, by (2.1), $w \geq \mathrm{wt}(\pi_{s(u)}(c)) = wt(u)/2 \geq l2^{k-l-1}$. So $t = w/2^{k-l-1} \geq l$.

□

2.2 Proof of Theorem 4

Proof. First, we show that for all $n, k, w \in \mathbb{N}$,

$$m(n,k,w) \leq 2^k - 2^{(k-1)-f_2(w)}.$$

Let $V \subseteq \mathbb{F}_2^n$ be dimension k subspace with $m(n,k,w)$ weight-w vectors. Let S be the set of nonzero, non-weight-w vectors in V. Let $b = f_2(w) + 2$. We claim that S intersects every dimension b subspace. Otherwise there is an equidistant linear code of dimension b in $V \setminus S$ and so by Proposition 2, w is divisible by $f_2(w)+1$, a contradiction. By Theorem 9, $|S| \geq 2^{k-b+1} - 1 = 2^{k-f_2(w)-1} - 1$, and hence the bound holds.

If the bound is met, then also by Theorem 9, S is the nonzero vectors of a dimension $k - f_2(w) - 1$ subspace of V. By Lemma 10, $w = t2^{f_2(w)}, t \geq k - f_2(w) - 1$, and $n \geq 2w - t + k - f_2(w) - 1$.

On the other hand, let $n, k, w \in \mathbb{N}$ and suppose that there exists an integer $t \geq (k-1) - f_2(w) \geq 0$ such that $w = t2^{f_2(w)}$ and $n \geq 2w - t + (k-1) - f_2(w)$. Set $l = (k-1) - f_2(w)$. Then $t \geq l$, $w = t2^{k-l-1}$, and $n \geq 2w - t + l$. By Lemma 10, there is a space $V \subseteq \mathbb{F}_2^n$ whose non-weight-w vectors have rank at most l. Thus $m(n,k,w) \geq 2^k - 2^l = 2^k - 2^{(k-1)-f_2(w)}$. Hence $m(n,k,w) = 2^k - 2^{(k-1)-f_2(w)}$.

□

3 Killers

3.1 Introduction

We now prove Theorem 7. This author and Lucas Sabalka found a more general theorem [6], but they used a different proof technique. Theorem 7 is a generalization of Theorem 6, the MacWilliams Extension Theorem. We also apply Theorem 7 to determine when $m(m,k,w) = 2^{k-1} - 1$ for odd w.

3.2 Binary c-killers

We prove the binary case separately because its proof more beautiful than the general case.

The *symmetric difference* of sets $S_1, S_2, \ldots, S_k \subseteq [n]$ is the set of elements of $[n]$ that occur in an odd number of S_i. We denote this by

$$\bigoplus_{j\in[k]} S_j = \{c \in [n] \,:\, |\{i \in [k] \,:\, c \in S_i\}| \equiv 1 (\text{mod } 2)\}\,.$$

Given $I \subseteq [k]$, denote

$$S(I) = \bigcap_{i\in I} S_i.$$

We have the following fact, similar to the principle of inclusion and exclusion.

Lemma 11.

$$\left|\bigoplus_{i\in[k]} S_i\right| = \sum_{\substack{I\subseteq[k]\\ I\neq\emptyset}} (-2)^{|I|-1}|S(I)|.$$

Proof. Given $x \in n$, define $C_x = \{i \in [k] : x \in S_i\}$.

$$\begin{aligned}
\sum_{\substack{I\subseteq[k]\\ I\neq\emptyset}} (-2)^{|I|-1}|S(I)| &= \sum_{\substack{I\subseteq[k]\\ I\neq\emptyset}} \sum_{x\in S(I)} (-2)^{|I|-1} \\
&= \sum_{x\in[n]} \sum_{\substack{I\subseteq C_x\\ I\neq\emptyset}} (-2)^{|I|-1} \\
&= \sum_{x\in[n]} -\frac{1}{2} \sum_{i=1}^{|C_x|} \binom{|C_x|}{i} (-2)^i \\
&= \sum_{x\in[n]} -\frac{1}{2} \left((-1)^{|C_x|} - 1\right) \\
&= \left|\bigoplus_{i\in[k]} S_i\right|
\end{aligned}$$

□

Let $S \subseteq \mathbb{F}^n$ and define $O(S)$ to be the size of the set of bit positions where all of the vectors of S overlap. More precisely,

$$O(S) = \left|\{i \in [n] : \pi_i(v) \neq 0 \,\forall\, v \in S\}\right|.$$

Lemma 12. *Let $n, k, c \in \mathbb{N}$, let $V, W \subseteq \mathbb{F}_2^n$ be subspaces, and let $\phi : V \to W$ be a c-killer. If B is a set of k linearly independent vectors from V then*

$$O(\phi(B)) = O(B) - c/2^{k-1}.$$

Proof. We proceed by induction on k. The case $k = 1$ is clear by the definition of a c-killer.

Let B be a set of $k > 1$ linearly independent vectors in $\mathbb{F}_2^n$. By Lemma 11,

$$\mathrm{wt}\left(\sum_{v\in B} v\right) = \sum_{\substack{I\subseteq B\\ I\neq\emptyset}} (-2)^{|I|-1} O(I) \tag{3.1}$$

and similarly

$$\mathrm{wt}\left(\sum_{v\in B} \phi(v)\right) = \sum_{\substack{I\subseteq B\\ I\neq\emptyset}} (-2)^{|I|-1} O(\phi(I)). \tag{3.2}$$

By induction and equations (3.1) and (3.2), we have

$$
\begin{aligned}
\mathrm{wt}\left(\sum_{v\in B}\phi(v)\right) &= \sum_{\substack{I\subseteq B\\ I\neq\emptyset}}(-2)^{|I|-1}O(\phi(I)) \\
&= (-2)^{|B|-1}O(\phi(B)) + \sum_{\substack{I\subseteq B\\ I\neq\emptyset,B}}(-2)^{|I|-1}O(\phi(I)) \\
&= (-2)^{|B|-1}O(\phi(B)) + \sum_{\substack{I\subseteq B\\ I\neq\emptyset,B}}(-2)^{|I|-1}(O(I)-c/2^{|I|-1}) \\
&= (-2)^{|B|-1}O(\phi(B)) + \sum_{\substack{I\subseteq B\\ I\neq\emptyset,B}}[(-2)^{|I|-1}O(I)+(-1)^{|I|}c] \\
&= (-2)^{|B|-1}O(\phi(B)) + \mathrm{wt}(\sum_{v\in B}v) - (-2)^{|B|-1}O(B) \\
&\quad + c[-1-(-1)^{|B|}]
\end{aligned}
$$

On the other hand,

$$
\mathrm{wt}\left(\sum_{v\in B}\phi(v)\right) = \mathrm{wt}(\phi(\sum_{v\in B}v)) = \mathrm{wt}(\sum_{v\in B}v) - c.
$$

Thus we have

$$
\begin{aligned}
\mathrm{wt}(\sum_{v\in B}v) - c &= (-2)^{|B|-1}O(\phi(B)) + \mathrm{wt}(\sum_{v\in B}v) - (-2)^{|B|-1}O(B) \\
&\quad + c[-1-(-1)^{|B|}].
\end{aligned}
$$

Cancelling $\mathrm{wt}(\sum_{v\in B}v) - c$ and rearranging, we have

$$
\begin{aligned}
(-2)^{|B|-1}O(\phi(B)) &= (-2)^{|B|-1}O(B) - c(-1)^{|B|} \\
O(\phi(B)) &= O(B) - c/2^{|B|-1}.
\end{aligned}
$$

□

Lemma 13. *Let $c > 1$ be an integer and let V and W be binary spaces. If $\phi : V \to W$ is a c-killer then there exists a code $\mathcal{C} \subseteq \mathbb{F}_2^{2c}$ with constant nonzero weight c and $\dim\mathcal{C} = \dim V$.*

Proof. Let B be a basis for V. By Lemma 12,

$$
O(B) - c/2^{|B|-1} = O(\phi(B)).
$$

In particular, $c/2^{|B|-1}$ is an integer. The lemma then follows from Proposition 2. □

We are now ready to prove our characterization of c-killers.

Proof of the binary case of Theorem 7. Let V,W be subspaces of $\mathbb{F}_2^n$, where $\dim V = k$, and suppose that $\phi : V \to W$ is a c-killer. If $c = 0$, then we are done by Theorem 6, the MacWilliams Extension Theorem. Thus we assume that $c \geq 1$.

By Lemma 13, there is a k-dimensional equidistant linear code, $\mathcal{C} \subseteq \mathbb{F}_2^{2c}$, whose nonzero weight is c. Since V and $\mathcal{C}$ are both k-dimensional vector spaces over $\mathbb{F}_2$, there is a linear bijection

$$\psi : V \to \mathcal{C}.$$

Define $W \times \mathcal{C} \subseteq \mathbb{F}_2^{n+2c}$ by

$$W \times \mathcal{C} = \{wv \,:\, w \in W, v \in \mathcal{C}\},$$

where wv is the vector formed by concatenating w and v. Consider

$$\phi \times \psi : V \to W \times \mathcal{C},$$

defined by

$$(\phi \times \psi)(v) = \phi(v)\psi(v).$$

Notice that $\mathrm{wt}((\phi \times \psi)(\vec{0})) = 0 = \mathrm{wt}(\vec{0})$. Moreover, given $v \in V \setminus \{\vec{0}\}$, we have

$$\mathrm{wt}((\phi \times \psi)(v)) = \mathrm{wt}(\phi(v)) + \mathrm{wt}(\psi(v)) = wt(v) - c + c = wt(v).$$

Thus $\phi \times \psi$ preserves weight. By the MacWilliams Extension Theorem, $\phi \times \psi$ is a coordinate permutation. But

$$\phi = \pi_{[n]} \circ (\phi \times \psi).$$

Thus ϕ is a coordinate permutation followed by a coordinate projection. □

3.3 General c-killers

We now prove the c-killer classification theorem for spaces over general finite fields.

Theorem 14 (Bonisoli [3])**.** *Let $n, k, w \in \mathbb{N}$. There exists a k-dimensional subspace $\mathcal{C} \subseteq \mathbb{F}_q^n$, all of whose nonzero vectors have weight w, if and only if there exists $t \in \mathbb{N}$ such that*

$$w = tq^{k-1}$$

and

$$n \geq t\frac{q^k - 1}{q - 1}.$$

Our proof of the general case for Theorem 7 will mirror the binary case. Let $\phi : V \to W$ be a c-killer. Set $k = \dim V$. We will establish that c is divisible by q^{k-1}. By Theorem 14, there is a k-dimensional equidistant linear code of weight c. We then "stitch" this code onto W, making ϕ a 0-killer. Finally we apply the MacWilliams Extension Theorem to determine that this new map is a monomial equivalence.

Given I, a multiset consisting of vectors from $\mathbb{F}_q^n$, we define the *common support* of I to be

$$\mathrm{cs}(I) = \{x \in [n] \,:\, \pi_x(v) \neq 0 \text{ for all } v \in I\}.$$

Given J, a multiset consisting of vectors from $\mathbb{F}_q^n$, we define the *zero sum set* of J to be

$$\mathrm{zs}(J) = \left\{ x \in [n] \,:\, \pi_x\left(\sum_{v \in J} v\right) = 0 \right\}.$$

Further, we define

$$O_J(I) = |\mathrm{cs}(I) \cap \mathrm{zs}(J)|.$$

In words, $O_J(I)$ is the number of coordinates in the common support of I where the sum of the vectors in J is 0. In particular, if $S = \{s\}$ then $O_\emptyset(S) = \mathrm{wt}(s)$ and $O_S(S) = 0$. The following lemma is not hard to prove.

Lemma 15. *Let $n \in \mathbb{N}$, and let S be a multiset of vectors in $\mathbb{F}_q^n$. Then*

$$\mathrm{wt}\left(\sum_{s \in S} s\right) = \sum_{\substack{I \subseteq S \\ I \neq \emptyset}} \sum_{J \subseteq I} (-1)^{|I|+|J|+1} O_J(I).$$

□

By using Lemma 15 and induction, we may prove the following in a manner very similar to the proof of Lemma 12.

Lemma 16. *Let $n, c \in \mathbb{N}$ and let $V, W \subseteq \mathbb{F}_q^n$ be subspaces. If $\phi : V \to W$ is a c-killer and $S \subseteq V$ is linearly independent then*

$$\sum_{J \subseteq S} (-1)^{|J|} O_J(S) = \sum_{J \subseteq S} (-1)^{|J|} O_{\phi(J)}(\phi(S)) + c. \tag{3.3}$$

□

The following is a generalization of Lemma 12.

Lemma 17. *With the setup in Lemma 16, we have*

$$O(S) = O(\phi(S)) + c\left(\frac{q-1}{q}\right)^{|S|-1}.$$

Proof. Denote $\mathbb{F}_q^\times = \mathbb{F}_q - \{0\}$ and denote the set of functions from S to $\mathbb{F}_q^\times$ by $\left(\mathbb{F}_q^\times\right)^S$. Let $\alpha = (\alpha_v)_{v \in S} \in \left(\mathbb{F}_q^\times\right)^S$. Let $J = \{v_1, \ldots, v_j\} \subseteq S$ and denote $\alpha J = \{\alpha_v v \,:\, v \in J\}$, and $\alpha \cdot J = \sum_{v \in J} \alpha_v v$.

For a fixed $\alpha \in \left(\mathbb{F}_q^\times\right)^{J\setminus\{v_j\}}$, we have

$$\begin{aligned}\sum_{\beta\in\mathbb{F}_q^\times} O_{(\alpha,\beta)J}(S) &= \sum_{\beta\in\mathbb{F}_q^\times} \sum_{\substack{x\in\mathrm{cs}(S)\\ \pi_x((\alpha,\beta)\cdot J)=0}} 1 \\ &= \sum_{x\in\mathrm{cs}(S)} \sum_{\substack{\beta\in\mathbb{F}_q^\times\\ \pi_x((\alpha,\beta)\cdot J)=0}} 1.\end{aligned}$$

Notice that for x in the common support of S, $\pi_x(v_j) \neq 0$ and hence if $\pi_x(\alpha\cdot(J\setminus\{v_j\})) \neq 0$ then there is exactly one nonzero solution, β, to $\pi_x((\alpha,\beta)\cdot J) = 0$. Otherwise there is no nonzero solution. Thus

$$\begin{aligned}\sum_{\beta\in\mathbb{F}_q^\times} O_{(\alpha,\beta)J}(S) &= \left|\{x\in\mathrm{cs}(S) \,:\, \pi_x(\alpha\cdot(J\setminus\{v_j\}))\neq 0\}\right| \\ &= O(S) - O_{\alpha(J\setminus\{v_j\})}(S).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{\gamma\in\left(\mathbb{F}_q^\times\right)^J} O_{\gamma J}(S) &= \sum_{\alpha\in\left(\mathbb{F}_q^\times\right)^{J\setminus\{v_j\}}} \sum_{\beta\in\mathbb{F}_q^\times} O_{(\alpha,\beta)J}(S) \\ &= \sum_{\alpha\in\left(\mathbb{F}_q^\times\right)^{J\setminus\{v_j\}}} \left[O(S) - O_{\alpha(J\setminus\{v_j\})}(S)\right] \\ &= (q-1)^{j-1}O(S) - \sum_{\alpha\in\left(\mathbb{F}_q^\times\right)^{J\setminus\{v_j\}}} O_{\alpha(J\setminus\{v_j\})}(S).\end{aligned}$$

Notice that the rightmost sum is in exactly the same form as the leftmost sum. Thus, by induction, we have

$$\begin{aligned}\sum_{\gamma\in\left(\mathbb{F}_q^\times\right)^J} O_{\gamma J}(S) &= \sum_{i=2}^{|J|} (-1)^{|J|+i}(q-1)^{i-1}O(S) \\ &= \frac{(q-1)}{q}\left[(q-1)^{|J|-1} + (-1)^{|J|}\right] O(S).\end{aligned}$$

Summing the left hand side of equation (3.3) over all $\alpha \in (\mathbb{F}_q^\times)^S$, we have

$$\begin{aligned}
&\sum_{\alpha\in(\mathbb{F}_q^\times)^S}\sum_{J\subseteq S}(-1)^{|J|}O_{\alpha J}(S)\\
&\qquad= \sum_{J\subseteq S}(-1)^{|J|}\sum_{\alpha\in(\mathbb{F}_q^\times)^S}O_{\alpha J}(S)\\
&\qquad= \sum_{J\subseteq S}(-1)^{|J|}\sum_{\beta\in(\mathbb{F}_q^\times)^{S\setminus J}}\sum_{\gamma\in(\mathbb{F}_q^\times)^J}O_{\gamma J}(S)\\
&\qquad= \sum_{J\subseteq S}(-1)^{|J|}(q-1)^{|S|-|J|}\frac{(q-1)}{q}\left[(q-1)^{|J|-1}+(-1)^{|J|}\right]O(S)\\
&\qquad= \frac{q-1}{q}O(S)\left[\sum_{J\subseteq S}(-1)^{|J|}(q-1)^{|S|-1}+\sum_{J\subseteq S}(q-1)^{|S|-|J|}\right]\\
&\qquad= (q-1)q^{|S|-1}O(S).
\end{aligned}$$

Thus, summing the entire equation (3.3), we get

$$(q-1)q^{|S|-1}O(S) = (q-1)q^{|S|-1}O(\phi(S)) + (q-1)^{|S|}c$$

and hence

$$O(S) = O(\phi(S)) + c\left(\frac{q-1}{q}\right)^{|S|-1}.$$

□

Thus if $\phi : V \to W$ is a c-killer and S is a basis for V then c is divisible by $q^{|S|-1}$. This proves Theorem 7 in general.

3.4 An Application of c-Killers

We will now apply the characterization of binary c-killers to determine the parameters for which w is odd and $m(n,k,w) = 2^{k-1} - 1$. We've already determined when $m(n,k,w) = 2^{k-1}$, so this is a next natural question.

Lemma 18. *Let $n,k,w \in \mathbb{N}$ where w is odd, $k \geq 1$, and $n \geq 1$. If there is a k-dimensional subspace $V \subseteq \mathbb{F}_2^n$ with exactly $2^{k-1}-1$ weight w vectors then one of the following properties holds.*

$$w = 1 \text{ and either a) } k = 1 \text{ and } n \geq 2 \text{ or b) } k = 2 \text{ or } 3 \text{ and } n \geq 3 \tag{3.4}$$

$$w \geq 3 \text{ and } k = 1 \tag{3.5}$$

$$w \geq 3 \text{ and } 2 \leq k \leq \lfloor \log_2 w \rfloor + 2 \text{ and } n \geq w + 2^{k-2} - 1 \tag{3.6}$$

$$w \geq 3 \text{ and } 2 \leq k \leq \lfloor \log_2 (w+1) \rfloor + 2 \text{ and } n \geq w + 2^{k-2}. \tag{3.7}$$

Proof. Suppose w is odd and there is a k-dimensional subspace $V < \mathbb{F}_2^n$ with $2^{k-1}-1$ weight w vectors. When $k \geq 2$, we define v to be the single odd weight vector of V that does not have weight w. Set $l = \mathrm{wt}(v)$. Without loss of generality,

$$v = \overbrace{1\dots1}^{R_1}\overbrace{0\dots0}^{R_0} \in V.$$

Here, R_i is the set of coordinates where v is i. In particular, $|R_1| = l$. Let $\mathcal{E}$ be the subspace of even weight vectors from V.

Case 1: $w = 1$.

Here we must have that $2^{k-1} \leq k$, and so $k = 1, 2$, or 3. The rest of (3.4) follows easily.

Case 2: $w \geq 3$ and $k = 1$.

In this case, (3.5) is satisfied.

Case 3: $w \geq 3$, $k \geq 2$, and $w > l$.

Let $e \in \mathcal{E} \setminus \{\vec{0}\}$ and notice that

$$\begin{aligned} w &= \mathrm{wt}(e+v) \\ &= \mathrm{wt}(\pi_{R_1}(e+v)) + \mathrm{wt}(\pi_{R_0}(e+v)) \\ &= l - \mathrm{wt}(\pi_{R_1}(e)) + \mathrm{wt}(\pi_{R_0}(e)), \end{aligned}$$

and thus

$$\mathrm{wt}(\pi_{R_1}(e)) = \mathrm{wt}(\pi_{R_0}(e)) - (w-l). \tag{3.8}$$

Since $wt(\pi_{R_0}(e)) \geq w - l$ for all $e \in \mathcal{E} \setminus \{\vec{0}\}$, we have that π_{R_0} is injective on $\mathcal{E}$. Thus we may define $\phi : \pi_{R_0}(\mathcal{E}) \to \pi_{R_1}(\mathcal{E})$ by

$$\phi = \pi_{R_1} \circ {\pi_{R_0}}^{-1}.$$

(ϕ simply assigns the right hand side of $e \in \mathcal{E}$ to its left hand side). By equation (3.8), ϕ is a $(w-l)$-killer. By Theorem 7 (the characterization of c-killers), there is a set of coordinates $S \subseteq R_0$ such that $\pi_S(\mathcal{E})$ is a equidistant linear code with nonzero weight $w-l$ and dimension equal to $\dim \pi_{R_0}(\mathcal{E}) = \dim \mathcal{E} = k-1$. Thus, by Proposition 2,

$$k - 1 \leq f_2(w-l) + 1 \leq \lfloor \log_2(w) \rfloor + 1.$$

Thus

$$k \leq \lfloor \log_2(w) \rfloor + 2.$$

Also by Proposition 2,

$$|R_0| \geq 2(w-l) - (w-l)/2^{k-2}. \tag{3.9}$$

Proposition 2 also tells us that 2^{k-2} divides $(w-l)$. Thus

$$2^{k-2} \leq w - 1.$$

Thus

$$l \leq w - 2^{k-2}. \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$\begin{aligned}
n &= |R_1| + |R_0| \\
&= l + |R_0| \\
&\geq l + 2(w-l) - (w-l)/2^{k-2} \\
&= 2w - w/2^{k-2} - l(1 - 1/2^{k-2}) \\
&\geq 2w - w/2^{k-2} - (w - 2^{k-2})(1 - 1/2^{k-2}) \\
&= w + 2^{k-2} - 1.
\end{aligned}$$

Thus (3.6) is satisfied.

Case 4: $w \geq 3$, $k \geq 2$, and $w < l$.

By applying arguments similar to those above, we find that (3.7) is satisfied. □

One can give constructions to show that the converse of Proposition 18 holds. We omit them here.

We now prove Proposition 8.

Proof of proposition 8. Suppose that $m(n,k,w) = 2^{k-1} - 1$. First we show that $k \leq 3$. Suppose to the contrary that $k > 3$. Since one of the clauses (3.4)-(3.7) must be satisfied and clauses (3.4) and (3.5) specify $k \leq 3$, it must be that (3.6) or (3.7) is satisfied.

We have $m(n,k,w) \neq 2^{k-1}$. Thus by Proposition 3, either $k > w+1$ or $n < w+k-1$. First consider $k > w+1$. Since one of (3.6) or (3.7) is true, it must be the case that

$$k \leq \max\{\lfloor \log_2 w \rfloor + 2, \lfloor \log_2 (w+1) \rfloor + 2\} = \lfloor \log_2 (w+1) \rfloor + 2,$$

and hence

$$w + 1 < k \leq \lfloor \log_2 (w+1) \rfloor + 2.$$

As it turns out, $w = 2$ is the largest w for which $w + 1 < \lfloor \log_2 (w+1) \rfloor + 2$. Thus

$$k \leq \lfloor \log_2 (2+1) \rfloor + 2 = 3.$$

On the other hand, suppose $n < w + k - 1$. Since one of (3.6) or (3.7) is true, we have

$$n \geq \min\left\{w + 2^{k-2} - 1, w + 2^{k-2}\right\} = w + 2^{k-2} - 1.$$

Thus,

$$w + 2^{k-2} - 1 \leq n < w + k - 1.$$

Therefore

$$2^{k-2} < k.$$

As it turns out, $k = 3$ is the largest k for which $2^{k-2} < k$.

We have established that $k \leq 3$. If $m(n, 1, w) = 0$ then n is not large enough to accommodate a single weight w vector. Thus $n < w$. This violates the assumption that $w \leq n$, so it is impossible to have $m(n, 1, w) = 0$.

If $m(n, 2, w) = 1$, then n is large enough to accommodate a weight w vector, but not two of them. Thus $n = w$. If $n = w$ then clearly $m(n, 2, w) = 1$.

If $m(n, 3, w) = 3$ then by Proposition 3, either $k > w+1$ or $n < w+k-1$. In the first case we have $3 > w + 1$ and hence $w < 2$. Thus $w = 1$. If $n < w + k - 1$ then $n < w + 2$, so $n \leq w + 1$. But $m(n, 3, w) > 1$ implies $n > w$. Thus $n = w + 1$.

On the other hand, if $w = 1$, then we may take

$$V = \left\{ v000\ldots000 \,:\, v \in \mathbb{F}_2^3 \right\}.$$

If $n = w + 1$, we may take the code V generated by the $3 \times n$ matrix

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 & 1 \end{bmatrix}.$$

□

4 Conjectures

4.1 Large Dimension

We have the following conjecture.

Conjecture 19. *Let $n, k, w \in \mathbb{N}$. If $n \geq k$ and $k \geq 2w$ then*

$$m(n, k, w) = \begin{cases} \binom{k+1}{w} & \text{if } w \text{ is even;} \\ \binom{k}{w} & \text{if } w \text{ is odd.} \end{cases}$$

According to [2], Khachatrian made the same conjecture in personal correspondence. We have checked it for n up to 14. Here we prove it for $n = k + 1$. Using a similar technique, it is possible to prove it for w odd and $n = k + 2$.

Proposition 20. *If $k, w \in \mathbb{N}$ and $k \geq 2w$ then*

$$m(k+1, k, w) = \begin{cases} \binom{k+1}{w} & w \text{ even;} \\ \binom{k}{w} & w \text{ odd.} \end{cases}$$

Proof. Suppose w is even. The span of all of the weight-w vectors in $\mathbb{F}_2^{k+1}$ has dimension k, so we're done. Thus we may assume that w is odd.

Given a vector space $W < \mathbb{F}_2^n$, define $A_w(W)$ to be the number of weight w vectors in W. Let $V < \mathbb{F}_2^{k+1}$ be k-dimensional with $A_w(V) = m(k+1, k, w)$. We have that V is monomially equivalent to a vector space with a generator matrix of the form

$$G = \left[\ I_k \;\middle|\; c\ \right].$$

Here c is a column vector. By permuting rows and columns of G we may assume that c is of the form

$$c = (\underbrace{0,0,0,\ldots,0,0,0}_{a},\underbrace{1,1,1,\ldots,1,1,1}_{b}).$$

Notice that $a+b=k$.

If we drop c from G, how many weight-w vectors are lost, and how many are gained? That is, are there more weight-w vectors in V or in $V' = \mathbb{F}_2^k \times \{0\}$, the code with generator matrix

$$G' = \left[\begin{array}{c|c} I_k & \vec{0} \end{array}\right]?$$

Let L be the *lost* vectors. That is,

$$L = V \setminus V'.$$

Let F be the *found* vectors. That is,

$$F = V' \setminus V.$$

We will construct an injective function $f : L \to F$. This will establish that $|L| \le |F|$ and thus,

$$A_w(V') = A_w(V) - |L| + |F| \ge A_w(V).$$

Set

$$B = \{a+1, a+2, \ldots, a+b=k\}.$$

Notice that for $v \in V$, we have

$$\mathrm{wt}(\pi_{B\cup\{k+1\}}(v)) \equiv 0 (\mathrm{mod}\ 2).$$

In particular, $b \ne k$, since this would imply that every vector in V has even weight. This is a contradiction, since w is odd, and $A_w(V) = m(n,k,w) > 0$. Note that

$$L = \left\{v \in V \,:\, \mathrm{wt}(\pi_{[k]}(v)) = w-1 \text{ and } \mathrm{wt}(\pi_B(v)) \equiv 1 (\mathrm{mod}\ 2)\right\}$$

and

$$F = \left\{v' \in V' \,:\, \mathrm{wt}(\pi_{[k]}(v')) = w \text{ and } \mathrm{wt}(\pi_B(v)) \equiv 1 (\mathrm{mod}\ 2)\right\}. \tag{4.1}$$

We will now define f. The definition will depend (slightly) on the parity of a.

Case 1: a is odd.

Given $l \in L$, define

$$g(l) = \min\left\{i \,:\, \mathrm{wt}\left(\overline{\pi_{[i]}(l)}\pi_{\{i+1,\ldots,k\}}(l)\right) = w\right\},$$

and set

$$f(l) = \overline{\pi_{[g(l)]}(l)}\pi_{\{g(l)+1,\ldots,k\}}(l)\,0.$$

In words: we scan across l from left to right, inverting bits one at a time. We stop when the weight on $[k]$ is w, and we change the last bit to 0. We have three things to show: that $g(l) < \infty$ (so that f is well-defined), that $f(l) \in F$, and that f is injective.

First we show that $g(l) < \infty$. In fact, $g(l) < k$. Notice that

$$\begin{aligned}\mathrm{wt}\left(\overline{\pi_{[k]}\left(l\right)}\right) &= k - \mathrm{wt}(\pi_{[k]}\left(l\right)) \\ &= k - (w-1) \\ &\geq 2w - (w-1) \\ &= w+1.\end{aligned}$$

We have that $\mathrm{wt}\left(\pi_{[k]}\left(l\right)\right) = w-1$, and $\mathrm{wt}\left(\overline{\pi_{[k]}\left(l\right)}\right) \geq w+1$. Since inverting a single bit in a vector changes its weight by one, one of the intermediate inversions considered in the definition of g must have weight w. Hence $g(l) < k$.

Now we show that $f(l) \in F$. By Equation (4.1), we need only show that $f(l) \in V'$, that $\mathrm{wt}(\pi_{[k]}\left(f(l)\right)) = w$, and $\mathrm{wt}(\pi_B\left(f(l)\right)) \equiv 1 (\mathrm{mod}\ 2)$. The only requirement for $f(l) \in V'$ is that $\pi_{k+1}\left(f(l)\right) = 0$. This is true by definition of f. By definition of g, it is clearly true that $\mathrm{wt}(\pi_{[k]}\left(f(l)\right)) = w$. It is left to show that $\mathrm{wt}(\pi_B\left(f(l)\right)) \equiv 1 (\mathrm{mod}\ 2)$. Either $g(l) \leq a$ or $g(l) > a$. In the first case,

$$\mathrm{wt}(\pi_B\left(f(l)\right)) = \mathrm{wt}(\pi_B\left(l\right)) \equiv 1 \bmod 2.$$

Consider $g(l) > a$. Inversion of a single bit changes the parity of the weight of a vector. Since $\mathrm{wt}(\pi_{[k]}\left(l\right)) = w-1$ and $\mathrm{wt}(\pi_{[k]}\left(f(l)\right)) = w$ (they have different parities), $g(l)$ must be odd. Since a is odd and f inverts all bits on $[a]$, $f(l)$ inverts an even number of bits on B. Thus

$$\mathrm{wt}(\pi_B\left(f(l)\right)) \equiv \mathrm{wt}(\pi_B\left(l\right)) \equiv 1 \bmod 2.$$

Finally, we show that f is injective. Let $l \in L$. We show how to construct l from $f(l)$. Given $m \in F$, define

$$g'(m) = \min\left\{i \,:\, \mathrm{wt}\left(\overline{\pi_{[i]}\left(m\right)}\pi_{\{i+1,\ldots,k\}}\left(m\right)\right) = w-1\right\},$$

and set

$$f'(m) = \overline{\pi_{[g(m)]}\left(m\right)}\pi_{\{g(m)+1,\ldots,k\}}\left(m\right)1.$$

Now, it is not necessarily the case that $g'(m) < \infty$. On the other hand, it is certainly the case that $g'(f(l)) \leq g(l)$, since

$$\begin{aligned}\mathrm{wt}\left(\overline{\pi_{[g(l)]}\left(f(l)\right)}\pi_{\{g(l)+1,\ldots,k\}}\left(f(l)\right)\right) &= \mathrm{wt}\left(\overline{\overline{\pi_{[g(l)]}\left(l\right)}}\pi_{\{g(l)+1,\ldots,k\}}\left(l\right)\right) \\ &= \mathrm{wt}\left(\pi_{[k]}\left(l\right)\right) \\ &= w-1.\end{aligned}$$

In fact, $g'(f(l)) = g(l)$. If $g'(f(l))$ were less than $g(l)$, then

$$\begin{aligned}
\mathrm{wt}\left(\pi_{[g'(f(l))]}\left(f(l)\right)\right) + \mathrm{wt}\left(\pi_{\{g'(f(l))+1,\ldots,k\}}\left(f(l)\right)\right)& \\
&= \mathrm{wt}\left(\pi_{[k]}\left(f(l)\right)\right) \\
&= w \\
&= (w-1)+1 \\
&= \mathrm{wt}\left(f'(f(l))\right)+1 \\
&= \mathrm{wt}\left(\overline{\pi_{[g'(f(l))]}\left(f(l)\right)}\right) + \mathrm{wt}\left(\pi_{\{g'(f(l))+1,\ldots,k\}}\left(f(l)\right)\right)+1.
\end{aligned}$$

Thus

$$\mathrm{wt}\left(\pi_{[g'(f(l))]}\left(f(l)\right)\right) = \mathrm{wt}\left(\overline{\pi_{[g'(f(l))]}\left(f(l)\right)}\right)+1.$$

But this implies

$$\begin{aligned}
\mathrm{wt}\left(\overline{\pi_{[g'(f(l))]}\left(l\right)}\pi_{\{g'(f(l))+1,\ldots,k\}}\left(l\right)\right)& \\
&= \mathrm{wt}\left(\overline{\pi_{[g'(f(l))]}\left(l\right)}\right) + \mathrm{wt}\left(\pi_{\{g'(f(l))+1,\ldots,k\}}\left(l\right)\right) \\
&= \mathrm{wt}\left(\pi_{[g'(f(l))]}\left(f(l)\right)\right) + \mathrm{wt}\left(\pi_{\{g'(f(l))+1,\ldots,k\}}\left(l\right)\right) \\
&= \mathrm{wt}\left(\overline{\pi_{[g'(f(l))]}\left(f(l)\right)}\right) + 1 + \mathrm{wt}\left(\pi_{\{g'(f(l))+1,\ldots,k\}}\left(l\right)\right) \\
&= \mathrm{wt}\left(\pi_{[g'(f(l))]}\left(l\right)\right) + 1 + \mathrm{wt}\left(\pi_{\{g'(f(l))+1,\ldots,k\}}\left(l\right)\right) \\
&= \mathrm{wt}(\pi_{[k]}\left(l\right))+1 \\
&= (w-1)+1 \\
&= w.
\end{aligned}$$

This contradicts the minimality of $g(l)$. We have established that $g'(f(l)) = g(l)$. Thus

$$\begin{aligned}
f'(f(l)) &= f'\left(\overline{\pi_{[g(l)]}\left(l\right)}\pi_{\{g(l)+1,\ldots,k\}}\left(l\right)0\right) \\
&= \overline{\overline{\pi_{[g(l)]}\left(l\right)}}\pi_{\{g(l)+1,\ldots,k\}}\left(l\right)1 \\
&= l.
\end{aligned}$$

Case 2: a is even.

This case is very similar to the case where a is odd, but we do not invert the first bit of $[a]$. □

4.2 A Complete Conjecture for Odd Weights

Conjecture 21. *If $n, k, w \in \mathbb{N}$ and w is odd then*

$$m(n,k,w) = M_{\mathbb{R}}(n,k,w).$$

□

We have checked the conjecture for n up to 14. Notice that by Theorem 1 (the formula for $M_{\mathbb{R}}(n,k,w)$), whenever we have been able to establish exact values for $m(n,k,w)$, they agree with $M_{\mathbb{R}}(n,k,w)$. In particular, suppose $k \leq w+1$ and $n \geq w+k-1$ (the conditions given in Proposition 3 that imply $m(n,k,w) = 2^{k-1}$). Either $w \leq n/2$ and $k-1 \leq w$, or $w > n/2$, in which case $n - w \leq n/2$, and since $n \geq w+k-1$, we have $k-1 \leq n-w$. Thus by Theorem 1,

$$m(n,k,w) = 2^{k-1} = M_{\mathbb{R}}(n,k,w).$$

Furthermore, for $k, w \in \mathbb{N}$ with $k \geq 2w$ and w odd, we have

$$\begin{aligned} m(k,k,w) = m(k+1,k,w) &= m(k+2,k,w) \\ &= \binom{k}{w} \\ = M_{\mathbb{R}}(k,k,w) = M_{\mathbb{R}}(k+1,k,w) &= M_{\mathbb{R}}(k+2,k,w). \end{aligned}$$

If w is odd and n is even then $n-w$ is odd. If Conjecture 21 is true then we would have

$$m(n,k,w) = M_{\mathbb{R}}(n,k,w) = M_{\mathbb{R}}(n,k,n-w) = m(n,k,n-w).$$

In fact, $m(n,k,w)$ does have this symmetry.

Proposition 22. *If $n, k, w \in \mathbb{N}$ where n is even and w is odd then*

$$m(n,k,w) = m(n,k,n-w).$$

Proof. Let B be a basis of odd weight vectors for $\mathcal{C}$ that achieves $A_w(\mathcal{C}) = m(n,k,w)$. Complement each element of B to get B'. Note that the code $\mathcal{C}'$ generated by B' has $A_{n-w}(\mathcal{C}') \geq A_w(\mathcal{C})$. Thus $m(n,k,n-w) \geq m(n,k,w)$. By symmetry, $m(n,k,n-w) = m(n,k,w)$. $\square$

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